Tricritical points of trails, their Euler digraphs and graphs: exact results on the Sierpinski gasket

1988 J. Phys. A: Math. Gen. 21 L903

(http://iopscience.iop.org/0305-4470/21/18/007)

J. Phys. A: Math. Gen. 21 (1988) L903–L909. Printed in the UK

LETTER TO THE EDITOR

Tricritical points of trails, their Euler digraphs and graphs: exact results on the Sierpinski gasket

I S Chang and Y Shapir

Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA

Received 17 May 1988

Abstract. The collapse transitions of trails, their oriented graphs and silhouettes (Eulerian digraphs and graphs, respectively), as the fugacity for crossings is increased, are investigated by exact decimation on the 2D Sierpinski gasget. Recursion relations between the generating functions for the three basic configurations on consecutive levels are derived. For all models we find tricritical points which move along a line in a four-dimensional parameter space, as the fugacity is varied, and terminate at a decoupled first- or second-order multicritical point. It suggests these models belong to distinct universality classes which differ from that of self-attracting polymer chains which do not undergo a collapse Θ transition on this fractal lattice.

Lattice models such as random walks (RW), self-avoiding walks (SAW) and self-attracting self-avoiding walks (SASAW) have been the focus of much attention [1]. They provide models for polymer chains in different regimes, and from the statistical mechanics viewpoint they serve as the generic examples to analyse scaling and fractal properties [2]. In the critical phenomena language the infinite RW is equivalent to a free (Gaussian) system at its critical point. The SAW is a critical $O(n)$ model with $n=0$ components [2]. The SASAW changes its behaviour from that related to a continuous phase transition (like the SAW) to a first-order phase transition into a collapse phase at low temperatures [2]. At the intermediate Flory Θ temperature its behaviour is described by a tricritical point of the $O(n)$, $n \to 0$ spin system [2–4]. In this regime the upper critical dimension changes from four to three and the 3D behaviour is Gaussian like a RW (but with logarithmic corrections) [5].

For a long time the Θ point of SASAW was the only known tricritical point of constrained walks. Recently new tricritical points were found in two other such models [6–9]: trails and their silhouettes. Trails [10] are lattice walks which are not allowed to step more than once on each bond but may self-cross through an already visited site (figure 1(c)). If only the shadows (figure 1(a)) of the trails are considered, the so-called silhouettes model is obtained [6] (they are the corresponding Euler graphs).

In the present letter we introduce yet another model interpolating between trails and silhouettes which we call digraphs. These are the oriented graphs obtained while keeping track of the trails' direction on the bonds, but discarding the chronological ordering in which the bonds were visited (figure 1(b)) [11]. The number of Euler trails associated with a given digraph is related to the number of spanning trees and may be expressed as the cofactor of a matrix simply related to the connectivity matrix. The problem of how many digraphs are associated with a given silhouette (Euler graph) is still unsolved [11]. The silhouettes and digraphs provide models for polymeric

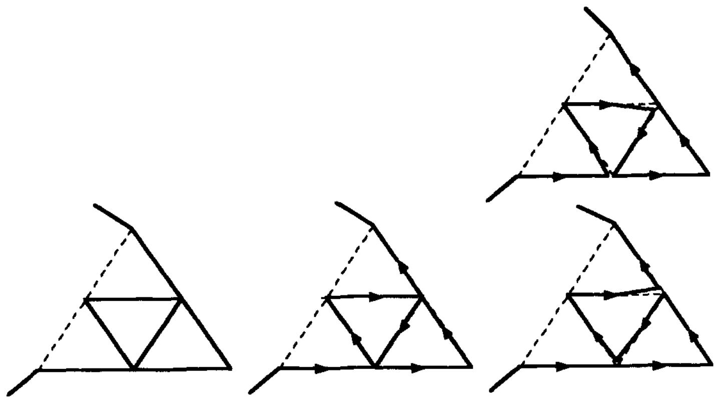

Figure 1. (*a*) A typical Eulerian graph (silhouettes) with two intersections. (*b*) One of three digraphs with the silhouette depicted in (*a*). (*c*) The two possible trails on the digraph in (*b*).

networks with tetrafunctional units formed by one type of monomer and by regular copolymers, respectively.

As suggested by Shapir and Oono [6], a fugacity w is associated with each intersection in order to achieve control over their averaged concentration. For a small concentration of intersections all these configurations share the long-distance scaling properties of SAW. For a large concentration (large fugacity) they will collapse into a compact phase. In between they may exhibit tricritical behaviours (similar to the Θ point). The questions of interest presently debated are whether each of these tricritical points is associated with a new universality class or whether part of them share the same one (or that of the Θ point) [6-9].

In the renormalisation group (RG) approach [6] near 4D, only the silhouettes exhibit a new tricritical fixed point of order $\varepsilon^{1/2}$ ($\varepsilon = 4-d$) [6]. So the possibility of a Θ point behaviour for trails and digraphs cannot be ruled out based only on momentum-space RG arguments. Series expansions in 3D indicate non-Gaussian behaviour for trails and silhouettes [7-9, 12]. They exhibit different exponents and for both models $\gamma_t < 1$ (for silhouettes: $\nu_t < \frac{1}{2}$ as well and both values of the exponents agree with the extrapolation of the $\varepsilon^{1/2}$ expansion to $\varepsilon = 1$ [9]). In 2D the exact exponents of the Θ point were conjectured recently [13]. For the trails, series expansions [7, 8, 12] and extensive MC calculations [14] indicate different values for some of the exponents. This is especially true for the crossover exponent ϕ and specific heat exponent α, since $\nu_t \geqslant \frac{1}{2}$ and $\gamma_t \geqslant 1$ for both trails and silhouettes. No results for ϕ and α of silhouettes in 2D are yet available. The configurations of digraphs (both in 2D and 3D) are only now being enumerated.

To shed more light on all these questions we have decided to explore their tricritical behaviour on the 2D Sierpinski gasket (SG) on which exact decimation is possible [15, 16]. Independent studies of SASAW on SG all agree that no collapsed phase and no Θ point exist on the 2D SG [17-20]. On the 3D SG the tricritical collapse occurs (as for branched polymers on both 2D and 3D SG [19, 20-22]).

We find tricritical collapses for trails, digraphs and silhouettes on the 2D SG which lends further support for them to belong to different universality classes, distinct from the Θ point. Although the results on the SG may hint at these conclusions and provide further insight into the relations between these models, some caution is in order. Indeed because of its special structure and its finite ramification the behaviour on this fractal is often not representative [15] as manifested, e.g., by the absence of the Θ point [19, 20–22].

The recursion relations on the 2D SG involve three restricted generating functions depicted in figure 2. On the n level of the iterative extension of the gasket they are as follows.

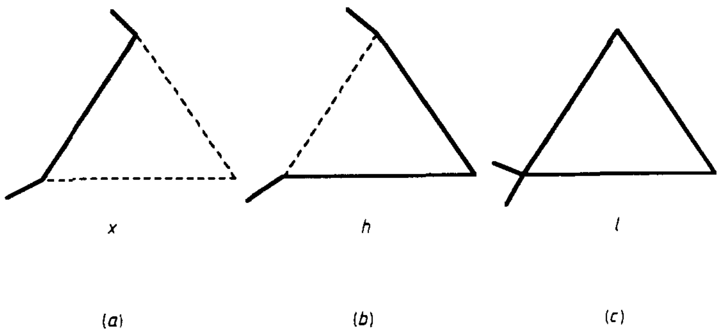

Figure 2. The three restricted generating functions.

(a) x_n, those trails (digraphs or graphs) going into the triangle by one apex and leaving from it by another apex without visiting the third apex (figure 2(a)).

(b) h_n, those which go into the triangle and leave it by a different apex but do visit the third one (figure 2(b)).

(c) l_n, those which go into the triangle and leave it through the same apex (figure 2(c)).

The recursion relations which relate $(x_{n+1}, h_{n+1}, l_{n+1})$ to (x_n, h_n, l_n) were derived by enumerating all possible configurations. We have assigned a local fugacity w for each *new* intersection (or loop, since on the SG it is equivalent) formed on the $(n+1)$ level not accounted for by the l_n of the n level. As an example we have depicted in figure 3 all configurations which contribute to h_{n+1}.

The recursion relations we obtain are

$$x_{n+1} = x_n^2 + 2x_nh_n + 2x_n^2h_n + h_n^2 + x_n^3 + Ah_n^2x_nw \tag{1a}$$

$$h_{n+1} = 2x_nh_n^2 + 2Bx_nh_nl_n + x_n^2h_n + Ch_n^2l_nw + Dh_n^3w \tag{1b}$$

$$l_{n+1} = h_n^3w + El_n^3w. \tag{1c}$$

The coefficients (A, B, C, D, E) change according to the models and are $(1, 1, 1, 1, 1)$ for silhouettes, $(2, 2, 3, 2, 6)$ for digraphs, and $(2, 2, 6, 2, 16)$ for the trails.

The recursion relations in (1) have been analysed and the general flow diagram is exhibited in figure 4 for the silhouettes model with $w = 1$. There are four different fixed points (FP).

(1) The SAW FP $(x^*, h^*, l^*) = (0.618, 0, 0)$ analysed in [17–22] with size exponent $\nu_{\mathrm{SAW}} = 0.799$.

(2) A discontinuity FP $(0, 0, 1)$ of the 'hard triangles' model on the SG, with $\nu = 1/\bar{d} = \ln 2/\ln 3 = 0.631$.

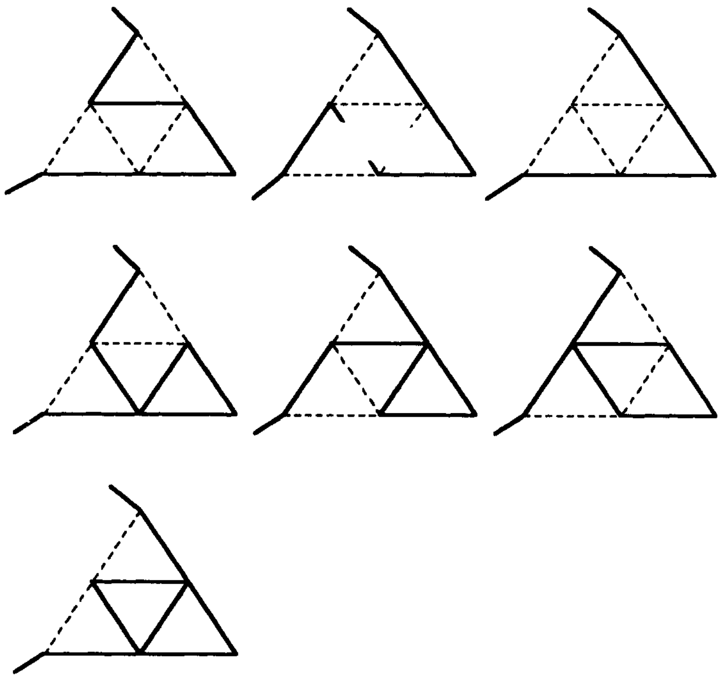

Figure 3. All configurations on the n level contributing to h_{n+1}.

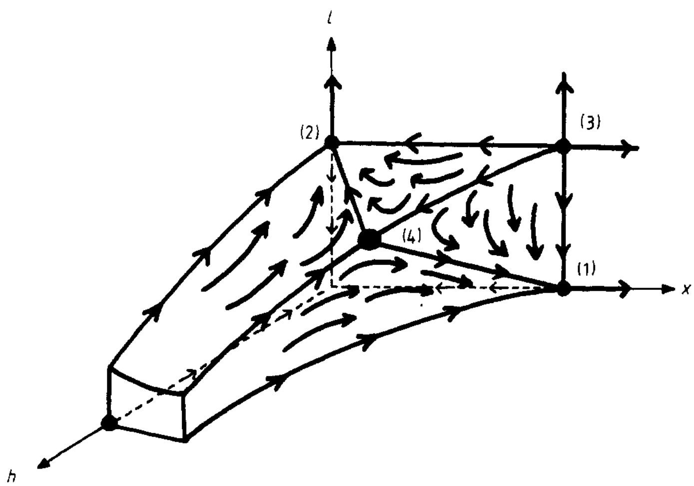

Figure 4. The flow diagram in the parameter space (for silhouettes at $w = 1$). The fixed points (1)-(4) are discussed in the text.

(3) A decoupled first- or second-order multicritical point (0.618, 0, 1), at which the above two behaviours of FP (1) and (2) occur simultaneously and independently. This FP is unstable in the direction perpendicular to the $h=0$ plane with a crossover exponent $\phi_h = 1.44$. This flow, for $h > 0$, is towards another fixed point.

(4) The tricritical FP (0.2960, 0.1819, 0.9969). This FP is unstable in two directions (has two relevant operators and two diverging length scales) and is stable in the third direction and is, therefore, a tricritical point. Linearisation of the recursion relations around this FP yields the eigenvalues: 2.99, 2.22 and 0.59. The eigenvector corresponding to the largest eigenvalue has non-trivial projections on all three axes and, therefore, controls the silhouettes' size dependence with the exponent:

$$\nu_g = \frac{\ln 2}{\ln 2.99} = 0.632 \tag{2}$$

(the subscript g stands for 'geometrical').

As expected this value is intermediate between the SAW swollen size exponent $\nu_{SAW} - 0.799$ and that of the compact phase $\nu_c = 1/\bar{d} \times 0.631$. We note that ν_g is quite close to the latter; a similar observation has been made for the tricritical point of branched polymers on the SG [20–22]. This is not surprising since the silhouettes model is also describing branched polymers with vertices of even degrees only (if the two ends are joined and close the path [6]). These configurations have the same size exponent as the 'open' silhouettes. The second positive exponent controls the divergence of the 'thermal correlations'

$$\begin{aligned} \xi_{th} &\approx |\Delta\varepsilon|^{-\nu_{th}} \\ \nu_{th} &= \frac{\ln 2}{\ln 2.22} = 0.869 \end{aligned} \tag{3}$$

where $|\Delta\varepsilon|$ is the distance from tricritical point (usually in the temperature direction but here the scaling field includes all three couplings because of the non-trivial eigenvectors). The crossover exponent at the tricritical point is

$$\phi = \nu_g / \nu_{th} = 0.727. \tag{4}$$

The third exponent in the direction of the irrelevant scaling field yields the exponent which controls the non-analytic leading-order corrections to scaling:

$$\Delta = \frac{-\nu_g}{\nu_{negative}} = \frac{\ln 0.59}{\ln 2.99} = 0.480. \tag{5}$$

As the local fugacity for intersections is increased to $w > 1$, FP (1) does not change and (2) and (3) change trivially to $(0, 0, w^{-1/2})$ and $(0.618, 0, w^{-1/2})$. However, due to non-linear coupling, the tricritical point drifts continuously nearer to the $h = 0$ plane. At the same time the tricritical exponents are varied. There is, therefore, a line of tricritical point as w is varied [19]. This unusual dependence of the scaling properties on a local parameters is certainly a manifestation of the specific geometric properties (finite ramification, etc) of the SG.

The line of fixed points ends at a critical value $w_c = 4$ at which the tricritical FP merges into the decoupled FP (3) in the $h = 0$ plane. For $w > w_c$ the decoupled FP (3) is stable and the continuous and discontinuous transitions are decoupled without any tricritical behaviour.

Table 1. The tricritical points and their exponents.

Model	w	x^*	h^*	l^*	ν_g	ϕ	Δ
Silhouettes	1	0.296	0.182	0.997	0.632	0.727	0.480
Digraphs	1	0.330	0.162	0.406	0.633	0.742	0.314
Trails	1	0.618	0	0.25	Decoupled fixed point ($\omega = \omega_c$)		
	0.5	0.406	0.131	0.352	0.631	0.756	0.217

The general features of the behaviour in digraphs is similar to that of silhouettes. Their tricritical properties for $w = 1$ are given in table 1 and their tricritical line merges to the decoupled point at $w_c = \frac{8}{3}$. More details will be published elsewhere.

For trails $w = 1$ is exactly the critical value at which the tricritical point (4) (which exists for $w < 1$) merges into the decoupled point. This is probably not a coincidence and it may be observed directly by looking at the recursion relation for h_n at the $w = 1$ decoupled point, which is

$$h_{n+1} = h_n + 2.736h_n^2 \tag{6}$$

namely, at $w = 1$ the decoupled point is marginally unstable. We therefore present in table 1 the tricritical properties of trails at $w = \frac{1}{2}$ as well.

To summarise, we have analysed the tricritical properties of trails, their digraphs and their silhouettes on the 2D SG. Contrary to the absent Θ point, we find tricritical points for all three models. This is consistent with each of them being in a distinct universality class at its collapse transition. More extensive results will be published in forthcoming works.

We acknowledge S Ramaswamy for valuable help in the first stages of this investigation, and A Guha and H A Lim for on-going discussions on these issues. The work is partially supported by a grant from the Eastman Kodak Company.

References

[1] Shlesinger M F and Weiss G H (ed) 1986 *The Wonderful World of Stochastics* (*Studies in Statistical Mechanics* **12**) (Amsterdam: North-Holland)
[2] de Gennes P G 1979 *Scaling Concepts in Polymer Physics* (Ithaca, NY: Cornell University Press)
[3] Flory P J 1971 *Principles of Polymer Chemistry* (Ithaca, NY: Cornell University Press); 1949 *J. Chem. Phys.* **17** 303
[4] de Gennes P G 1975 *J. Physique* **36** L55
[5] Duplantier B 1982 *J. Physique* **43** 991
[6] Shapir Y and Oono Y 1984 *J. Phys. A: Math. Gen.* **17** 39
[7] Lim H A, Guha A and Shapir Y 1988 *J. Phys. A: Math. Gen.* **21** 773
[8] Guha A, Lim H A and Shapir Y 1988 *J. Phys. A: Math. Gen.* **21** 1043
[9] Shapir Y, Guha A and Lim H A 1988 in preparation
[10] Malakis A 1976 *J. Phys. A: Math. Gen.* **9** 1283
[11] Harary F and Palmer E M 1973 *Graphical Enumeration* (New York: Academic)
[12] Chang I S, Guha A, Lim H A and Shapir Y 1988 *J. Phys. A: Math. Gen.* **21** L559
[13] Duplantier B and Saleur H 1987 *Phys. Rev. Lett.* **59** 539
[14] Meirovitch H and Lim H A 1988 *Phys. Rev.* A submitted
[15] Gefen Y, Aharony A, Shapir Y and Mandelbrot B B 1984 *J. Phys. A: Math. Gen.* **17** 435
[16] Dhar D 1978 *J. Math. Phys.* **19** 5

[17] Klein D J and Seitz W A 1984 *J. Physique* **45** L241
[18] Rammal R, Toulouse G and Vannimenus J 1984 *J. Physique* **45** 389
[19] Dhar D and Vannimenus J 1987 *J. Phys. A: Math. Gen.* **20** 199
[20] Knezevic M and Vannimenus J 1986 *Phys. Rev. Lett.* **56** 1591
[21] Knezevic M and Vannimenus J 1987 *Phys. Rev.* B **35** 4988
[22] Knezevic M and Vannimenus J 1987 *J. Phys. A: Math. Gen.* **20** L969